

Inequalities for quantum skew information

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Abstract

We study quantum information inequalities and show that the basic inequality between the quantum variance and the metric adjusted skew information generates all the multi-operator matrix inequalities or Robertson type determinant inequalities studied by a number of authors. We introduce an order relation on the set of functions representing quantum Fisher information that renders the set into a lattice with an involution. This order structure generates new inequalities for the metric adjusted skew informations. In particular, the Wigner-Yanase skew information is the maximal skew information with respect to this order structure in the set of Wigner-Yanase-Dyson skew informations.

Key words and phrases: Quantum covariance, metric adjusted skew information, Robertson-type uncertainty principle, operator monotone function, Wigner-Yanase-Dyson skew information.

1 Introduction

Recently there has been a lot of effort going into the investigation of various quantum information inequalities. We shall present a unified view that will cover most of the already known results and add some more. The notion of metric adjusted skew information was introduced by the third author [9] as a measure of quantum information generalizing the Wigner-Yanase-Dyson skew informations [18]. The metric adjusted skew information is a convex function on the manifold of states and also satisfies other requirements, suggested by Wigner and Yanase, for a measure of the information content of

a state with respect to a conserved observable. These requirements include additivity with respect to the aggregation of isolated subsystems and time independence for an isolated system. One more requirement, super additivity, fails spectacularly [7] for all the measures, including the Wigner-Yanase skew information.

We introduce an order relation \preceq on the set of functions representing quantum Fisher information that renders the set into a lattice with a maximal and a minimal element. The order relation induces an order relation on the set of metric adjusted skew informations that renders also this set into a lattice with the SLD-information as maximal element. There is no minimal element.

The Wigner-Yanase-Dyson skew informations given by

$$I_\rho(p, A) = -\frac{1}{2}\text{Tr}[\rho^p, A][\rho^{1-p}, A] \quad 0 < p < 1$$

are shown to be increasing in the parameter p with respect to this new order relation in the interval $(0, 1/2]$ and decreasing in the interval $[1/2, 1)$. There is thus maximum in the Wigner-Yanase skew information for $p = 1/2$. More elementary, the function $p \rightarrow I_\rho(p, A)$ is increasing in $(0, 1/2]$ and decreasing in $[1/2, 1)$ for fixed ρ and A .

1.1 Basic notations and definitions

The metric adjusted skew information is defined by setting

$$(1) \quad I_\rho^c(A) = \frac{m(c)}{2}\text{Tr} i[\rho, A^*]c(L_\rho, R_\rho)i[\rho, A]$$

for every positive definite $n \times n$ density matrix ρ and every $n \times n$ matrix A . The function c of two variables is a so called regular Morozova-Chentsov function which may be written on the form

$$c(x, y) = \frac{1}{yf(xy^{-1})} \quad x, y > 0,$$

where f is an operator monotone function defined on the positive half-axis with $f(1) = 1$ satisfying the functional equation $f(t) = tf(t^{-1})$ for $t > 0$. The regularity condition means that $m(c) = f(0) > 0$. The operators L_ρ and R_ρ are the commuting positive definite left- and right- multiplication operators

by ρ . If we want to emphasize the dependence of the representing operator monotone function f we may also denote the metric adjusted skew information by $I_\rho^f(A)$. The metric adjusted skew information is thus proportional to the metric length, as measured by the quantum Fisher information, of the commutator $i[\rho, A]$. In the case of the Wigner-Yanase-Dyson information this proportionality was already noticed in [16]. The constant is however important as it can be chosen such that the metric adjusted skew information coincides with the variance on pure states.

We have tacitly extended the definition of the metric adjusted skew information to include the case where A may not be self-adjoint. This does not directly make sense in physical applications, but it turns out to be a useful mathematical tool. The symmetry of c implies that

$$(2) \quad I_\rho^c(A + iB) = I_\rho^c(A) + I_\rho^c(B)$$

for self-adjoint A and B . In particular, the metric adjusted skew information is convex in ρ also for non self-adjoint “observables”.

1.2 The dynamical uncertainty principle

The basic quantum information inequality is given by

$$(3) \quad 0 \leq I_\rho^c(A) \leq \text{Var}_\rho(A),$$

where the symmetrized variance

$$\text{Var}_\rho(A) = \frac{1}{2} \text{Tr} \rho(A^*A + AA^*) - |(\text{Tr} \rho A)|^2.$$

The inequality was stated and proved in [9] for self-adjoint A by noting that the metric adjusted skew information is a convex function in ρ while the covariance is concave, and that the two measures coincide on pure states. Gibilisco et al. [2, Proposition 9.2] subsequently gave another proof. The inequality generalizes an earlier result by Luo for the Wigner-Yanase skew information [13]. Since the symmetrized variance satisfies

$$\text{Var}_\rho(A + iB) = \text{Var}_\rho(A) + \text{Var}_\rho(B)$$

for self-adjoint A and B , we immediately obtain the inequality (3) also for non self-adjoint A .

The above observations effectively reduce the multi-dimensional versions of the quantum information inequalities given by a number of authors [3] to the main inequality (3). Let (A_1, \dots, A_k) be a tuple of (possibly non self-adjoint) matrices. We then obtain the inequality

$$(4) \quad \left(I_\rho(A_i, A_j) \right)_{i,j=1}^k \leq \left(\text{Cov}_\rho(A_i, A_j) \right)_{i,j=1}^k$$

where

$$I_\rho^c(A, B) = \frac{m(c)}{2} \text{Tr } i[\rho, A^*] c(L_\rho, R_\rho) i[\rho, B]$$

and

$$\text{Cov}_\rho(A, B) = \frac{1}{2} \text{Tr } \rho(A^*B + BA^*) - (\text{Tr } \rho A^*)(\text{Tr } \rho B).$$

Indeed, we only have to notice the identity

$$I_\rho^c(\xi_1 A_1 + \dots + \xi_k A_k) = \left(\left(I_\rho(A_i, A_j) \right)_{i,j=1}^k \xi \mid \xi \right), \quad \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ x_k \end{pmatrix},$$

for arbitrary complex numbers ξ_1, \dots, ξ_k , and the similar identity for the covariance matrix. This reduces the matrix version (4) to the basic quantum information inequality (3). In particular, we obtain the determinant inequality

$$(5) \quad 0 \leq \det \left(I_\rho(A_i, A_j) \right)_{i,j=1}^k \leq \det \left(\text{Cov}_\rho(A_i, A_j) \right)_{i,j=1}^k$$

by the well known formula $\det A = \exp \text{Tr } \log A$, since the logarithm is operator monotone¹. This version has been coined the “dynamic uncertainty principle”.

2 Inequalities for quantum skew information

We introduce various methods to compare measures of quantum information for one observable. The generalization to several observables is then obtained as explained in the introduction.

¹In fact, the trace function $A \rightarrow \text{Tr } f(A)$ is increasing for any increasing function f .

The metric adjusted skew information may be written [9, Proposition 3.4] on the form

$$(6) \quad I_\rho^c(A) = \text{Tr } \rho A^2 - \frac{m(c)}{2} \text{Tr } A d_c(L_\rho, R_\rho) A,$$

where

$$d_c(x, y) = \frac{x + y}{m(c)} - (x - y)^2 c(x, y) \quad x, y > 0$$

is operator concave. Since d_c is homogeneous of degree one we may write

$$\frac{m(c)}{2} d_c(x, y) = y \tilde{f}(xy^{-1}),$$

where the function

$$\tilde{f}(t) = \frac{f(0)}{2} d_c(t, 1) = \frac{t + 1}{2} - (t - 1)^2 \frac{f(0)}{2f(t)}.$$

Since \tilde{f} is operator concave and defined on the positive half-axis it is also operator monotone [8]. With this notation the metric adjusted skew information takes the form

$$(7) \quad I_\rho^c(A) = \text{Tr } \rho A^2 - \text{Tr } A R_\rho \tilde{f}(L_\rho R_\rho^{-1}) A$$

as it is studied in [14].

We begin by providing a more detailed classification of the representing functions for quantum fisher information.

Theorem 2.1. *Let $f: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be a function satisfying*

- (i) *f is operator monotone,*
- (ii) *$f(t) = t f(t^{-1})$ for all $t > 0$,*
- (iii) *$f(1) = 1$.*

Then f admits a canonical representation

$$(8) \quad f(t) = \frac{1 + t}{2} \exp \int_0^1 \frac{(\lambda^2 - 1)(1 - t)^2}{(\lambda + t)(1 + \lambda t)(1 + \lambda)^2} h(\lambda) d\lambda$$

where the weight function $h: [0, 1] \rightarrow [0, 1]$ is measurable. The equivalence class containing h is uniquely determined by f . Any function on the given form maps the positive half-axis into itself and satisfy the conditions (i), (ii) and (iii).

Proof. The third author [6, Theorem 1] gave an exponential representation of the functions $f: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ satisfying (i) and (ii) of the form

$$(9) \quad f(t) = e^\beta \frac{1+t}{\sqrt{2}} \exp \int_0^1 \frac{\lambda^2 - 1}{\lambda^2 + 1} \cdot \frac{1+t^2}{(\lambda+t)(1+\lambda t)} h(\lambda) d\lambda$$

where $h: [0, 1] \rightarrow [0, 1]$ is measurable and $\exp \beta = f(i) \exp(-i\pi/4)$. Setting

$$f(1) = e^\beta \frac{2}{\sqrt{2}} \exp \int_0^1 \frac{\lambda^2 - 1}{\lambda^2 + 1} \cdot \frac{2}{(\lambda+1)^2} h(\lambda) d\lambda = 1$$

and solving for $\exp \beta$ we obtain

$$f(t) = \frac{1+t}{2} \exp \int_0^1 \frac{\lambda^2 - 1}{\lambda^2 + 1} \left(\frac{1+t^2}{(\lambda+t)(1+\lambda t)} - \frac{2}{(\lambda+1)^2} \right) h(\lambda) d\lambda$$

which reduces to the expression in the theorem. The uniqueness of h (up to equivalence) and the sufficiency of the condition follows from the reference.

QED

Following several authors we denote by \mathcal{F}_{op} the set of functions characterized in Theorem 2.1. A function $f \in \mathcal{F}_{\text{op}}$ is said to be regular if its extension to the closed positive half line satisfies $f(0) > 0$. If $f \in \mathcal{F}_{\text{op}}$ is regular we set

$$(10) \quad \check{f}(t) = \frac{f(0)}{f(t)} \quad \text{and} \quad \check{c}(x, y) = y^{-1} \check{f}(xy^{-1})$$

and may write

$$I_\rho^f(A) = \frac{1}{2} \text{Tr } i[\rho, A^*] \check{c}(L_\rho, R_\rho) i[\rho, A].$$

Notice that \check{c} is a symmetric function in two positive variables, and that the metric adjusted skew information $I_\rho^f(A)$ is increasing in the transform \check{f} , cf. also [3, 4, 5].

The representing function $f_p \in \mathcal{F}_{\text{op}}$ of the Wigner-Yanase-Dyson skew information $I_\rho(p, A)$ with parameter p is given by

$$f_p(t) = p(1-p) \cdot \frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)} \quad 0 < p < 1.$$

We observe that the transform

$$\check{f}_p(t) = \frac{(t^p-1)(t^{1-p}-1)}{(t-1)^2} = \frac{t+1-(t^p+t^{1-p})}{(t-1)^2}$$

is increasing in $p \in (0, 1/2]$ and decreasing in $p \in [1/2, 1)$. It follows that the Wigner-Yanase-Dyson skew information $I_\rho(p, A)$, for fixed ρ and A , is an increasing function of p in the interval $(0, 1/2]$ and a decreasing function of p in the interval $[1/2, 1)$ with maximum in the Wigner-Yanase skew information.

Proposition 2.2. *The transform \check{f} of a regular function $f \in \mathcal{F}_{op}$ has a canonical representation*

$$(11) \quad \check{f}(t) = \frac{f(0)}{f(t)} = \frac{1}{(1+t)} \exp \int_0^1 \frac{t(\lambda^2 - 1)}{\lambda(\lambda + t)(1 + \lambda t)} h(\lambda) d\lambda,$$

where the weight function $h : [0, 1] \rightarrow [0, 1]$ is measurable and

$$\int_0^1 \frac{h(\lambda)}{\lambda} d\lambda < \infty.$$

The equivalence class containing h is uniquely determined by f . Any function on the given form is the transform \check{f} of a regular function f in \mathcal{F}_{op} .

Proof. The integrability condition for a weight function h of a regular function $f \in \mathcal{F}_{op}$ follows from Theorem 2.1. Furthermore,

$$f(0) = \frac{1}{2} \exp \int_0^1 \frac{(\lambda^2 - 1)}{\lambda(1 + \lambda)^2} h(\lambda) d\lambda$$

and thus

$$\frac{f(0)}{f(t)} = \frac{1}{(1+t)} \exp \int_0^1 \left(\frac{(\lambda^2 - 1)}{\lambda(1 + \lambda)^2} - \frac{(\lambda^2 - 1)(1 - t)^2}{(\lambda + t)(1 + \lambda t)(1 + \lambda)^2} \right) h(\lambda) d\lambda$$

which reduces to the expression in the proposition. **QED**

Since the integrand in (11) is non-positive, we realize that \check{f} and thus the metric adjusted skew information is decreasing in the weight function h .

Definition 2.3. *Let $f, g \in \mathcal{F}_{op}$ and set*

$$\varphi(t) = \frac{t+1}{2} \frac{f(t)}{g(t)} \quad t > 0.$$

We write $f \preceq g$ if the function $\varphi \in \mathcal{F}_{op}$.

The function φ obviously satisfy $\varphi(1) = 1$ and $\varphi(t) = t\varphi(t^{-1})$ for $t > 0$. The condition in the definition is thus equivalent to operator monotonicity of φ . Setting

$$f_{\min}(t) = \frac{2t}{t+1} \quad \text{and} \quad f_{\max}(t) = \frac{1+t}{2} \quad t > 0,$$

it is known that $f_{\min} \leq f \leq f_{\max}$ for every function $f \in \mathcal{F}_{\text{op}}$. But we realize that also

$$(12) \quad f_{\min} \preceq f \preceq f_{\max} \quad \text{for every } f \in \mathcal{F}_{\text{op}}.$$

This is so since $f_{\min} \preceq f$ is reduced to operator monotonicity of the function $t \rightarrow tf(t)^{-1}$, while $f \preceq f_{\max}$ is reduced to operator monotonicity of f itself. Since as noted $\varphi \leq f_{\max}$ for $\varphi \in \mathcal{F}_{\text{op}}$ we realize that

$$(13) \quad f \preceq g \quad \Rightarrow \quad f \leq g \quad \text{for } f, g \in \mathcal{F}_{\text{op}}.$$

In particular, $f \preceq g$ and $g \preceq f$ implies $f = g$.

Theorem 2.4. *The relation \preceq is a partial order relation on \mathcal{F}_{op} rendering $(\mathcal{F}_{\text{op}}, \preceq)$ into a lattice with f_{\min} as the minimal element and f_{\max} as the maximal element. Furthermore, if $f, g \in \mathcal{F}_{\text{op}}$ then*

$$f \preceq g \quad \Leftrightarrow \quad h_f \geq h_g \quad \text{a.e.},$$

where h_f and h_g respectively are representing functions of f and g as in Theorem 2.1.

Proof. The function φ in Definition 2.3 has the representation

$$\varphi(t) = \frac{1+t}{2} \exp \int_0^1 \frac{(\lambda^2 - 1)(1-t)^2}{(\lambda+t)(1+\lambda t)(1+\lambda)^2} (h_f(\lambda) - h_g(\lambda)) d\lambda$$

and is therefore operator monotone, according to Theorem 2.1, if and only if $0 \leq h_f(\lambda) - h_g(\lambda) \leq 1$ for almost all $\lambda \in [0, 1]$. Therefore, $f \preceq g$ if and only if $h_f \geq h_g$ almost everywhere.

It follows that \preceq is an order relation. Indeed, if $f_1 \preceq f_2$ and $f_2 \preceq f_3$ then, for representing functions, $h_{f_1} \geq h_{f_2}$ and $h_{f_2} \geq h_{f_3}$ almost everywhere. Consequently, $h_{f_1} \geq h_{f_3}$ almost everywhere and thus $f_1 \preceq f_3$.

For arbitrary $f, g \in \mathcal{F}_{\text{op}}$ with representing functions h_f and h_g we define $f \wedge g$ as the function in \mathcal{F}_{op} with representing function $\max\{h_f, h_g\}$. Similarly,

we define $f \vee g$ as the function in \mathcal{F}_{op} with representing function $\min\{h_f, h_g\}$. It follows that

$$f \wedge g \preceq f \preceq f \vee g \quad \text{and} \quad f \wedge g \preceq g \preceq f \vee g.$$

If furthermore $\psi \preceq f$ and $\psi \preceq g$ for a $\psi \in \mathcal{F}_{\text{op}}$ it follows that $\psi \preceq f \wedge g$. If similarly $f \preceq \psi$ and $g \preceq \psi$ it follows that $f \vee g \preceq \psi$. Therefore $(\mathcal{F}_{\text{op}}, \preceq)$ is a lattice. **QED**

Notice that for $f, g \in \mathcal{F}_{\text{op}}$ we have the relations

$$f \wedge g \leq \min\{f, g\} \leq f \leq \max\{f, g\} \leq f \vee g,$$

with a similar statement for g .

We next show that the lattice \mathcal{F}_{op} is equipped with a natural involution.

Definition 2.5. *We define for $f \in \mathcal{F}_{\text{op}}$ the function*

$$(14) \quad f^\sharp(t) = \frac{t}{f(t)} \quad t > 0.$$

It follows from the general theory of operator monotone functions [8, 2.6. Corollary] that f^\sharp is operator monotone, and since obviously $f^\sharp(1) = 1$ and $f^\sharp(t) = t f^\sharp(t^{-1})$ we realize that also $f^\sharp \in \mathcal{F}_{\text{op}}$.

It also follows that $f^{\sharp\sharp} = f$ and that $f(t) = t^{1/2}$ is the unique fixpoint of the involution $f \rightarrow f^\sharp$ in \mathcal{F}_{op} . Furthermore, $f \preceq g$ implies $g^\sharp \preceq f^\sharp$ for functions $f, g \in \mathcal{F}_{\text{op}}$. We notice that f^\sharp is not regular for a regular $f \in \mathcal{F}_{\text{op}}$. All of these assertions may also be verified by calculating the representing weight functions.

Proposition 2.6. *Let $f \in \mathcal{F}_{\text{op}}$ with representing weight function h as in Theorem 2.1. Then $f^\sharp \in \mathcal{F}_{\text{op}}$ with representing weight function $1 - h$.*

Proof. We first write

$$f^\sharp(t) = \frac{t}{f(t)} = \frac{2t}{t+1} \exp \left[- \int_0^1 \frac{(\lambda^2 - 1)(1-t)^2}{(\lambda+t)(1+\lambda t)(1+\lambda)^2} h(\lambda) d\lambda \right]$$

and since $f_{\min}(t) = 2t(t+1)^{-1}$ has 1 as representing function, we obtain

$$f^\sharp(t) = \frac{t+1}{2} \exp \int_0^1 \frac{(\lambda^2 - 1)(1-t)^2}{(\lambda+t)(1+\lambda t)(1+\lambda)^2} (1 - h(\lambda)) d\lambda$$

and the assertion is proved. **QED**

We obtain from Proposition 2.2 and the preceding remarks:

Corollary 2.7. *The restriction of the order relation \preceq to the regular part of \mathcal{F}_{op} induces an order relation on the set of metric adjusted skew informations.*

2.1 Optimality of the Wigner-Yanase information

The order relation on the set of metric adjusted skew informations introduced in Corollary 2.7 is tractable in the sense that we only have to study and compare the representing functions of the associated quantum Fisher informations as they are given in Theorem 2.1.

The function $f_p \in \mathcal{F}_{op}$ representing the Wigner-Yanase-Dyson skew information with parameter $p \in (0, 1)$ has weight function

$$h_p(\lambda) = \frac{1}{\pi} \arctan \frac{(\lambda^p + \lambda^{1-p}) \sin p\pi}{1 - \lambda - (\lambda^p - \lambda^{1-p}) \cos p\pi} \quad 0 < \lambda < 1,$$

according to the representation given in Theorem 2.1, cf. [9, Theorem 2.7].

Theorem 2.8. *The functions $h_p(\lambda)$ are decreasing in $p \in (0, 1/2]$ for any λ with $0 < \lambda < 1$.*

Proof. We consider a fixed $\lambda \in (0, 1)$, set $z_0 = -\lambda + i\varepsilon$ for a small $\varepsilon > 0$, and obtain

$$h_p(\lambda) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} f_\varepsilon(p),$$

where

$$\begin{aligned} f_\varepsilon(p) &= \arg((1 - z_0^p)(1 - z_0^{1-p})) \\ &= \arg(1 - z_0^p) + \arg(1 - z_0^{1-p}). \end{aligned}$$

Obviously, we have $f_\varepsilon(p) = f_\varepsilon(1-p)$. Thus, to show that $f_\varepsilon(p)$ is increasing over the left half $[0, 1/2]$ of the domain, it suffices to show that the first term $\arg(1 - z_0^p)$ is concave in p over the interval $[0, 1]$.

We can show that the function $g(p) = \arg(1 - z^p)$ is concave over $[0, 1]$ for any z in the “lune” $\mathcal{I} = \{z : |z| < 1, \Im z > 0\}$, including z_0 for sufficiently small $\varepsilon > 0$. Since $\arg = \Im \log$, the second derivative $g''(p)$ is given by

$$g''(p) = \Im \frac{-z^p (\log z)^2}{(1 - z^p)^2} = \frac{1}{p^2} \Im \frac{-z^p (\log z^p)^2}{(1 - z^p)^2}$$

and we have to show that this is non-positive for $z \in \mathcal{I}$ and $0 \leq p \leq 1$. In fact, it is enough to do this for $p = 1$ only, since for $z \in \mathcal{I}$ and $0 \leq p \leq 1$, then $z^p \in \mathcal{I}$ too.

The imaginary part of a complex number is non-positive if and only its complex argument is between π and 2π . Thus we need to show

$$q(z) = \arg \frac{-z(\log z)^2}{(1-z)^2} \in [\pi, 2\pi] \quad \forall z \in \mathcal{I}.$$

Putting $z = r \exp(i\theta)$, with $0 < r \leq 1$ and $0 \leq \theta \leq \pi$,

$$\begin{aligned} q(z) &= \arg(-z) + 2 \arg \log z - 2 \arg(1-z) \\ &= \pi + \theta + 2 \arctan \frac{\theta}{\log r} + 2 \arctan \frac{r \sin \theta}{1 - r \cos \theta}. \end{aligned}$$

For $\theta = 0$ (z real) $q(z)$ is obviously π . For $\theta = \pi$,

$$q(z) = 2\pi + 2 \arctan \frac{\pi}{\log r} < 2\pi$$

for $0 < r < 1$. We can now show that for fixed r , $q(z)$ increases with θ , which implies that q is indeed between π and 2π for $z \in \mathcal{I}$. The first derivative is

$$\frac{\partial q}{\partial \theta} = 2 \frac{\log r}{(\log r)^2 + \theta^2} + \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}.$$

Because of the inequality $\cos \theta \geq 1 - \theta^2/2$, we obtain a lower bound

$$\begin{aligned} \frac{\partial q}{\partial \theta} &\geq \frac{1 - r^2}{(1 - r)^2 + r\theta^2} + 2 \frac{\log r}{(\log r)^2 + \theta^2} \\ &= \frac{\phi(r) \log r + \theta^2 \psi(r)}{((1 - r)^2 + r\theta^2)((\log r)^2 + \theta^2)} \end{aligned}$$

where

$$\begin{aligned} \phi(r) &= (1 - r^2) \log r + 2(1 - r)^2, \\ \psi(r) &= 1 - r^2 + 2r \log r. \end{aligned}$$

The first derivative $\psi'(r) = 2(1 - r + \log r)$ and the second derivative $\psi''(r) = 2(1/r - 1)$ is positive. Therefore ψ' is increasing and since $\psi'(1) = 0$, we derive

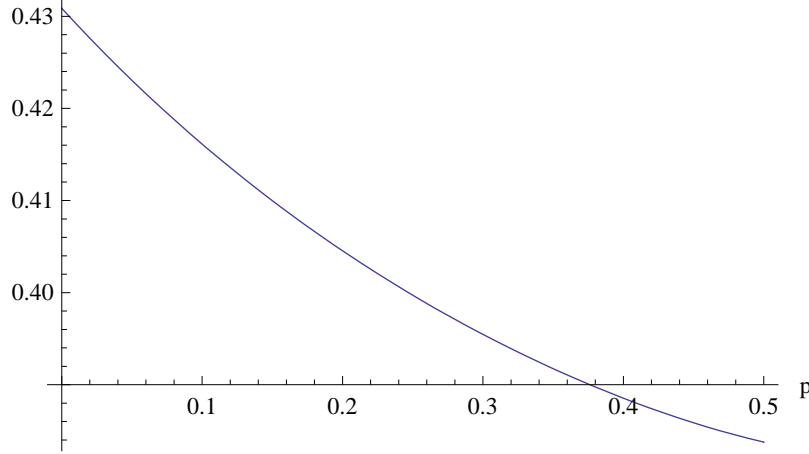


Figure 1: $p \rightarrow h_p(\lambda)$ for $\lambda = 1/2$

that ψ' is negative and thus ψ is decreasing. But since $\psi(1) = 0$, we derive that $\psi(r) \geq 0$ for $0 < r < 1$. We furthermore calculate

$$\begin{aligned}\phi'(r) &= -2r \log r + \frac{1}{r} + 3r - 4, \\ \phi''(r) &= \frac{1}{r^2}(r^2 - 2r^2 \log r - 1).\end{aligned}$$

The parenthesis in the expression for $\phi''(r)$ has derivative $-4r \log r \geq 0$ and value zero in $r = 1$. It is thus non-positive, so ϕ is concave and ϕ' therefore decreasing. Since $\phi'(1) = 0$ we derive that $\phi' \geq 0$, so ϕ is increasing and since $\phi(1) = 0$, we conclude that ϕ is non-positive. We have consequently proved

$$\frac{\partial q}{\partial \theta} \geq 0$$

for $0 < \theta < \pi$ and $0 < r < 1$, and the proof is complete. **QED**

Theorem 2.8 is illustrated by the graph in Figure 1. The graphs for different $\lambda \in (0, 1)$ look very much alike except for a change in units. An equivalent formulation of the result is that the function

$$\varphi(t) = \frac{t+1}{2} \frac{f_p(t)}{f_q(t)} = \frac{p(1-p)}{2q(1-q)} \cdot \frac{(t+1)(t^q-1)(t^{1-q}-1)}{(t^p-1)(t^{1-p}-1)} \quad t > 0.$$

is operator monotone (and thus belongs to \mathcal{F}_{op}) for $0 < p \leq q \leq 1/2$.

We have thus proved that $f_p \preceq f_q$ for $0 < p \leq q \leq 1/2$ and $f_p \succeq f_q$ for $1/2 \leq p \leq q < 1$. The Wigner-Yanase skew information is therefore the maximal element among the Wigner-Yanase-Dyson informations with respect to the order relation inherited from $(\mathcal{F}_{\text{op}}, \preceq)$.

Another example is the variant bridge considered in [9] with representing functions $f_p \in \mathcal{F}_{\text{op}}$ with weight functions according to the representation in Theorem 2.1 given by

$$h_p(\lambda) = \begin{cases} 0, & \lambda < 1 - p \\ p, & \lambda \geq 1 - p \end{cases} \quad 0 \leq p \leq 1.$$

This family of weight functions is decreasing in the parameter $p \in [0, 1]$. The corresponding quantum Fisher informations are therefore increasing in the parameter p and connects the SLD-metric for $p = 0$ with the Bures metric for $p = 1$, and they are regular for $p < 1$. The corresponding metric adjusted skew informations, defined for $p \in [0, 1)$, are increasing with respect to the order relation \preceq .

References

- [1] A. Andai. Uncertainty principle with quantum Fisher information. *J. Math. Phys.* 49, 012106, 2008.
- [2] P. Gibilisco, D. Imparato and T. Isola. Uncertainty principle and quantum Fisher information. *J. Math. Phys.* 48, 072109, 2007.
- [3] P. Gibilisco, D. Imparato and T. Isola. A Robertson-type uncertainty principle and quantum Fisher information. *arXiv: math-ph/ 0707.1231v1*, 2007.
- [4] P. Gibilisco, D. Imparato and T. Isola. Inequalities for quantum Fisher information. *arXiv: math-ph/ 0702058*, 2007.
- [5] P. Gibilisco, F. Hiai and D. Petz. Quantum covariance, quantum Fisher information and the uncertainty principle. *arXiv: math-ph/ 0712.1208*, 2007.
- [6] F. Hansen. Characterizations of symmetric monotone metrics on the state space of quantum systems. *Quantum Information and Computation*, 6: 597–605, 2006.

- [7] F. Hansen. The Wigner-Yanase entropy is not subadditive. *Journal of Statistical Physics*, 126: 643–648, 2007.
- [8] F. Hansen and G.K. Pedersen. Jensen’s inequality for operators and Löwner’s theorem. *Mathematische Annalen* 258: 229-241, 1982.
- [9] F. Hansen. Metric adjusted skew information. *arXiv: math-ph/ 0607049*, to appear in *Proc. Natl. Acad. Sci. U.S.A.*
- [10] H. Kosaki. Matrix trace inequalities related to uncertainty principle. *Internat. J. Math.*, 16(6): 629-645, 2005.
- [11] S.L. Luo, Q. Zhang. On skew information. *IEEE Trans. Inform. Theory* 50(8): 1778-1782, 2004.
- [12] S.L. Luo, Q. Zhang. Correction to ‘On skew information’. *IEEE Trans. Inform. Theory* 51(12): 4432, 2005.
- [13] S.L. Luo. Wigner-Yanase skew information and uncertainty relations. *Phys. Rev. Lett.*, 91: 180403, 2003.
- [14] D. Petz, V.E. Sándor Szabó. From quasi-entropy to skew information. *arXiv: math/ 0712.2881*, 2007.
- [15] D. Petz. Monotone metrics on matrix spaces. *Linear Algebra Appl.* 244: 81-96, 1996.
- [16] D. Petz. and Hasegawa. On the Riemannian metric of α -entropies of density matrices. *Lett. Math. Phys.*, 38: 221–22, 1996.
- [17] H.P. Robertson. An indeterminacy relation for several observables and its classical interpretation. *Phys. Rev.* 46: 794-801, 1934.
- [18] E.P. Wigner and M.M. Yanase. Information contents of distributions. *Proc. Natl. Acad. Sci. U.S.A.*, 49: 910–918, 1963.

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